DSP: Lec 4

Quiz

1. Prove that

$$Z\left[a^{n} x(n)\right] = X\left(\frac{z}{a}\right)$$

2. Deduce the z-transform for

a) $x(n-2)$ b) $x(n+1)$

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$$x(n) \longrightarrow X(z) = Z[x(n)]$$

$$Z[x(n)] = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$Z\left[a^{n} x(n)\right] = \sum_{n=0}^{\infty} a^{n} x(n)\, z^{-n}$$

$$= x(0) + a\,x(1)\, z^{-1} + a^{2} x(2)\, z^{-2} + \cdots$$

$$= x(0) + x(1)\left(\frac{z}{a}\right)^{-1} + x(2)\left(\frac{z}{a}\right)^{-2} + \cdots$$

$$x(n) \longrightarrow X(z) = x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + \cdots$$

$$\therefore X\left(\frac{z}{a}\right) = x(0) + x(1)\left(\frac{z}{a}\right)^{-1} \cdots$$

$$= Z\left[a^{n} x(n)\right]$$

$\boxed{2}$

a) $$Z\left[x(n-2)\right] = \sum_{n=0}^{\infty} x(n-2)\, z^{-n}$$

$$= x(-2) + x(-1)\, z^{-1} + x(0)\, z^{-2} \cdots$$

$$= x(-2) + x(-1)\, z^{-1} + z^{-2}\underbrace{\left[x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} \cdots\right]}_{X(z)}$$

$$= x(-2) + x(-1)\, z^{-1} + z^{-2}\, X(z)$$

b) $$Z\left[x(n+1)\right] = \sum_{n=0}^{\infty} x(n+1)\, z^{-n}$$

$$= x(1) + x(2)\, z^{-1} + x(3)\, z^{-2} + x(4)\, z^{-3} \cdots$$

$$= z \cdot z^{-1}\left[x(1) + x(2)\, z^{-1} \cdots\right]$$

$$= z\left[x(1)\, z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3} \cdots\right]$$

$$= z\left[\underbrace{x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} \cdots}_{X(z)} - x(0)\right]$$

$$= z\left[X(z) - x(0)\right] = z\, X(z) - z\, x(0)$$

c) $Z\left[x(n+2)\right]$

$$= \sum_{n=0}^{\infty} x(n+2)\, z^{-n}$$

$$= x(2) + x(3) z^{-1} + x(4) z^{-2} + \cdots$$

$$= z^{2} \cdot z^{-2}\left[x(2) + x(3) z^{-1} + x(4) z^{-2} + \cdots\right]$$

$$= z^{2}\left[x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4} + \cdots\right]$$

$$= z^{2}\left[x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + \cdots - x(0) - x(1) z^{-1}\right]$$

$$= z^{2}\left[X(z) - x(0) - x(1) z^{-1}\right]$$

$$\boxed{Z\left[x(n+2)\right] = z^{2} X(z) - x(0) z^{2} - x(1) z}$$

* shifting Property:-

$\rightarrow$ Delay $\quad x(n-a)$

$\rightarrow$ Advance $\quad x(n+a)$

Find Z.T for $x(n-1)$ $\boxed{\text{Delay}}$

$$x(n) \xrightarrow{Z.T} Z[x(n)] = X(z) = \sum_{n=0}^{\infty} x(n) z^{-1}$$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} \cdots$$

$$Z[x(n-1)] = \sum_{n=0}^{\infty} x(n-1) z^{-1}$$

$$= x(-1) + x(0) z^{-1} + x(1) z^{-2} + \cdots$$

$$= x(-1) + z^{-1}\left[x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots\right]$$

$$= x(-1) + z^{-1} x(z)$$

$$Z[x(n-2)] = \sum_{n=0}^{\infty} x(n-2) z^{-n}$$

$$= x(-2) + x(-1) z^{-1} + x(0) z^{-2}$$

$$= x(-2) + x(-1) z^{-1} + z^{-2}\left[x(0) + x(1) z^{-1} + \cdots\right]$$

$$\boxed{Z[x(n-2)] = x(-2) + x(-1)z^{-1} + z^{-2}X(z)}$$

$$x(n) \xrightarrow{Z.T} X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

assume $x(0) = 0, \quad n < 0$

$x(-1), x(-2), x(-3) \cdots\cdots = 0$

$$\therefore Z[x(n-2)] = z^{-2} X(z)$$

$$Z[x(n-1)] = z^{-1} X(z)$$

$$\therefore Z[x(n-m)] = z^{-m} X(z)$$

$$; X(z) = Z[x(n)]$$

Delay

$$x(n) \longrightarrow \boxed{z^{-2}} \longrightarrow x(n-2)$$

$$x(n) \longrightarrow \boxed{z^{-m}} \longrightarrow x(n-m)$$

$\boxed{\text{Advance}}$

Find Z.T for $x(n+1)$:

$$Z[x(n+1)] = \sum_{n=0}^{\infty} x(n+1) z^{-n}$$

$$= x(1) + x(2) z^{-1} + x(3) z^{-2} + \cdots$$

$$= z \cdot z^{-1} [x(1) + x(2) z^{-1} + \cdots]$$

$$= z [x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + \cdots]$$

$$= z [x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots - x(0)]$$

$$= z [X(z) - x(0)] = z X(z) - z x(0)$$

$$* \; Z[x(n+1)] = z X(z) - z x(0)$$

$$Z[x(n+2)] = \sum_{n=0}^{\infty} x(n+2) z^{-n}$$

$$= \underline{x(2) + x(3) z^{-1} + x(4) z^{-2} \cdots}$$

$$= z^{2} \cdot z^{-2} [\qquad\qquad\qquad]$$

$$= Z^{2}\left[x(2)\,z^{-2} + x(3)\,z^{-3} + \cdots\cdots\right]$$

$$= Z^{2}\left[x(0) + x(1)\,z^{-1} + x(2)\,z^{-2} + x(3)\,z^{-3} + \cdots \cdots - x(0) - x(1)\,z^{-1}\right]$$

$$= Z^{2}\left[X(z) - x(0) - x(1)\,z^{-1}\right]$$

$$\boxed{Z\left[x(n+2)\right] = z^{2}\,X(z) - x(0)\,z^{2} - x(1)\,z}$$

$$x(n) \longrightarrow \boxed{Z^{m}} \longrightarrow x(n+m)$$

في حالة ال Differen

← اللى هيجيبه الدكتور في الإمتحان هو ال (Delay)

$$\boxed{Z\left[x(n-m)\right] = Z^{-m}\cdot X(z)}$$

دى في حالة ال Difference eqn.

* initial value

$$x(0) = x(n)\Big|_{n=0} = \lim_{z \to \infty} x(z)$$

* final value

$$x(\infty) = x(n)\Big|_{n=\infty} = \lim_{z \to 1} (z-1)\, x(z)$$

$$= \lim_{z \to 1} (1 - z^{-1})\, x(z)$$

$(1 - z^{-1})$ هى نفسها $\frac{z-1}{z}$ فى الحالة الأولى مكتوبة

$(Z-1)$ بس لأن المقام عبارة عن Z و $(Z \leftarrow 1)$ فلا تؤثر.

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* Inverse Z-T $[Z^{-1}T]$

$$x(n) \xrightarrow{Z.T} x(z)$$

discrete time domain $\xleftarrow[\text{Inverse}]{Z^{-1}.T}$ z. domain

طرق الـ $[Z^{-1}T]$ ←

$\boxed{1}$ using [P.F] (closed form solution)

② using long division (open form solution)

open form ← يجيبلك sequence
$x(0), x(1), x(2), \ldots$

closed ← يجيب $x(n)$

$\boxed{Ex}$

$$X(z) = \frac{1}{1 - 1.5 z^{-1} + 0.5 z^{-1}}$$

find $x(n)$.

solu. using P.F

$$X(z) = \frac{z^2}{z^2 - 1.5z + 0.5}$$

$$z^2 - 1.5z + 0.5 = (z - 1)(z - 0.5)$$

$$x(z) = Z\left[\frac{Z}{(Z-1)(Z-0.5)}\right]$$

$$= Z\left(\frac{A_1}{Z-1} + \frac{A_2}{Z-0.5}\right)$$

$$A_1 = \frac{1}{1-0.5} = 2 \qquad A_2 = \frac{0.5}{-0.5} = -1$$

$$x(z) = Z\left[\frac{2}{Z-1} - \frac{1}{Z-0.5}\right]$$

$$x(z) = \frac{2Z}{Z-1} - \frac{Z}{Z-0.5}$$

$$Z^{-1}T \Rightarrow x(n) = 2\,u(n) - (0.5)^n$$

$$x(n) = 2\,u(n) - (0.5)^n$$

← لو لم نكتب $u(n)$ صح ولو كتبتها صح
دورها إنها تعرفني فيه (shift) ولا لأ.

$\boxed{Ex}$

$$x(z) = \frac{1}{(z-1)(z-0.3)}$$

using Partial fraction

$$x(z) = \frac{A_1}{z-1} + \frac{A_2}{z-0.3}$$

$$A_1 = \frac{1}{0.7} = 1.4 \qquad A_2 = \frac{-1}{0.7} = -1.4$$

$$x(z) = \frac{1.4 * z * z^{-1}}{z-1} - \frac{1.4 * z * z^{-1}}{z-0.3}$$

z^{-1} بتعمل (shift)

$$z^{-1}T \Rightarrow x(n) = 1.4\, u(n-1) - 1.4\,(0.3)^{n-1}\, u(n-1)$$

دورها ال (shift)

وتؤكد لنا المالة $(0.3)^n$ ليس لها وجود إلا بعد القيمة $n=1$

$$n=0 \Rightarrow x(0) = 0$$

$$x(0) = x(n)\Big|_{n=0} = \lim_{z\to\infty} x(z)$$

$$= \lim_{z\to\infty} \frac{1}{(z-1)(z-0.3)} = \frac{1}{\infty} = 0$$

ممكن تجيبها بأى قانون منهم $x(0)$

$\boxed{Ex}$ $\boxed{\text{Midterm Previous year}}$

$$x(z) = \frac{z^3}{(z-1)(z^2-z+1)}$$

① Find $x(0), x(1), x(2)$, using long division method.

② Find $x(n)$ using P.F.

↳ or Find $x(0), x(1), x(2)$ using P.F

هيا نفسها هتجيب $x(n)$ وتحسب منها قيم $x(0)$ ، $x(1)$ ، $x(2)$

sol

$$(z-1)(z^2-z+1) = z^3 - z^2 + z - z^2 + z - 1$$

$$x(z) = \frac{z^3}{z^3 - 2z^2 + 2z - 1}$$

← القسمة المطولة

طلب منك $x(0), x(1), x(2)$ يبقى هتجيب الثلاثة دول و تقف

$$1 + 2z^{-1} + 2z^{-2}$$

$$z^3 - 2z^2 + 2z - 1 \;\big)\; z^3$$

$$\frac{-z^3 \pm 2z^2 \mp 2z \pm 1}{2z^2 - 2z + 1}$$

$$\frac{-2z^2 \pm 4z \mp 4 \pm 2z^{-1}}{2z - 3 + 2z^{-1}}$$

$$X(z) = 1 + 2z^{-1} + 2z^{-2} + \cdots$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

$$\boxed{x(0) = 1 \;,\; x(1) = 2 \;,\; x(2) = 2}$$

$\boxed{2}$ using P.F

$$X(z) = \frac{z^3}{(z-1)(z^2 - z + 1)}$$

$$= z\left[\frac{z^2}{(z-1)(z^2 - z + 1)}\right]$$

$$X(z) = Z\left[\frac{A}{Z-1} + \frac{BZ+C}{Z^2-Z+1}\right]$$

$$A = \frac{1}{1-1+1} = 1$$

الحد الثاني هنوحد المقامات ونقارن المعاملات .

$B = 0 \quad , \quad C = 1$

$$X(z) = Z\left[\frac{1}{Z-1} + \frac{1}{Z^2-Z+1}\right]$$

$$= \frac{Z}{Z-1} + \frac{Z}{Z^2-Z+1}$$

$$\sin(wn) \xrightarrow{Z.T} \frac{Z\sin(w)}{Z^2-2Z\cos(w)+1}$$

$$\cos(wn) \longrightarrow \frac{Z(Z-\cos w)}{Z^2-2Z\cos w+1}$$

$$\frac{\sin(w)\,Z}{(Z^2-Z+1)*\sin w} \Rightarrow \frac{Z\sin w}{Z^2-2Z\cos w+1}$$

$$2\cos(w) = 1 \Rightarrow \cos(w) = \frac{1}{2}$$

$$\therefore w = 60^\circ = \frac{\pi}{3} \text{ rad}$$

$$\sin w = \sin 60 = \frac{\sqrt{3}}{2}$$

$$\frac{Z\,(\sqrt{3}/2)}{\frac{\sqrt{3}}{2}\left[Z^2 - Z + 1\right]} \longrightarrow \sin(wn) \text{ or } \sin\left(\frac{\pi}{3}n\right)$$

$$\Downarrow Z^{-1}T$$

$$= \frac{1}{\sqrt{3}/2}\sin\left(\frac{\pi}{3}n\right) = \frac{2}{\sqrt{3}}\sin\left(\frac{\pi}{3}n\right)$$

$$\boxed{Z^{-1}T \Rightarrow x(n) = u(n) + \frac{2}{\sqrt{3}}\sin\left(\frac{\pi}{3}n\right)}$$

← إذا طلب $x(0)$ ، $x(1)$ ، $x(2)$ ⇐ نعوض عن n بالقيم دي.

Ex $y(n) - y(n-1) + 0.25\, y(n-2) = u(n)$

For unit step i/p :

① find $y(0), y(1), y(2)$ using long division method.

② Find the unit step response $y(n)$ using p.f.

Sol

Z.T $Y(z) - z^{-1} Y(z) + 0.25 z^{-2} Y(z) = \frac{z}{z-1}$

where $u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$\therefore u(z) = \frac{z}{z-1}$

$(1 - z^{-1} + 0.25 z^{-2}) Y(z) = \frac{z}{z-1}$

$$Y(z) = \frac{z}{(z-1)(1 - z^{-1} + 0.25 z^{-2})} \quad * \frac{z^2}{z^2}$$

$$Y(z) = \frac{z^3}{(z-1)(z^2 - z + 0.25)}$$

$$(Z-1)(Z^2 - Z + 0.25) = Z^3 - Z^2 + 0.25Z - Z^2 + Z - 0.25$$

$$= Z^3 - 2Z^2 + 1.25Z - 0.25$$

<u>using long division</u>

$$Y(z) = \frac{Z^3}{Z^3 - 2Z^2 + 1.25Z - 0.25}$$

$$1 + 2Z^{-1} + 2.75Z^{-2}$$

$$Z^3 - 2Z^2 + 1.25Z - 0.25 \;\big)\; Z^3$$

$$\underline{-Z^3 \pm 2Z^2 \mp 1.25Z \pm 0.25}$$

$$2Z^2 - 1.25Z + 0.25$$

$$\underline{-2Z^2 \pm 4Z \mp 2.5 \pm 0.5Z^{-1}}$$

$$+2.75Z - 2.25 + 0.5Z^{-1}$$

$$y(0) = y(n)\big|_{n=0} = 1$$

$$y(1) = y(n)\big|_{n=1} = 2$$

$$y(2) = y(n)\big|_{n=2} = 2.75$$

$\boxed{2}$ using P.F

ده اللى بنجيبله P.F فهو اللى بنعوض فيه

$$y(z) = \frac{z^3}{(z-1)(z^2 - z + 0.25)}$$

$$= \frac{z^3}{(z-1)(z-0.5)^2} = z\left[\frac{z^2}{(z-1)(z-0.5)^2}\right]$$

$$= z\left[\frac{A_1}{z-1} + \frac{A_2}{(z-0.5)^2} + \frac{A_3}{(z-0.5)}\right]$$

$$A_1 = \frac{1}{(0.5)^2} = 4 \quad , \quad A_2 = \frac{(0.5)^2}{(-0.5)} = -0.5$$

* To find A_3

Put $z = 0$

حط أى قيمة لـ z بحيث لا تجعل المقام = صفر

← وقارن الأصل بال P.F بتاعه

$$0 = -A_1 + 4A_2 - 2A_3 \Rightarrow A_3 = -3$$

$$\frac{z^2}{(z-1)(z-0.5)^2} = \frac{A_1}{z-1} + \frac{A_2}{(z-0.5)^2} + \frac{A_3}{(z-0.5)}$$

$$Y(z) = \frac{4Z}{Z-1} - \frac{0.5Z}{(Z-0.5)^2} - \frac{3Z}{(Z-0.5)}$$

$$n \xrightarrow{Z.T} \frac{Z}{(Z-1)^2}$$

$$\frac{Z}{(Z-0.5)^2} = \frac{Z}{(0.5)^2\left(\frac{Z}{0.5}-1\right)^2} = \frac{Z}{(0.25)\left(\frac{Z}{0.5}-1\right)^2}$$

$$\cancel{d} = \frac{0.5 \cdot \overbrace{Z/0.5}^{\to a}}{(0.25)\left(\underbrace{\frac{Z}{0.5}}_{a}-1\right)^2}$$

$$a^n x(n) \xrightarrow{Z.T} X\left(\frac{Z}{a}\right)$$

$$\boxed{\therefore x(n) = 4u(n) - \cancel{A\left(\frac{Z}{0.5}\right)}\ (0.5)^n * n - 3(0.5)^n}$$

$$a^n n \xrightarrow{Z.T} \left.\frac{Z}{(Z-1)^2}\right|_{Z=\frac{Z}{a}}$$

$$= \frac{Z/a}{\left(\frac{Z}{a}-1\right)^2} = \frac{aZ}{(Z-a)^2}$$

Final value

$y(\infty) = 4$

$\boxed{EX}$ $X(z) = \dfrac{4Z}{Z^2 + 0.1Z + 0.2}$

Find $x(n)$

$\boxed{Sol}$

$$\sin(wn) \xrightarrow{Z.T} \frac{Z \sin(w)}{Z^2 - 2Z\cos(w) + 1}$$

$$a^n \sin(wn) \longrightarrow \frac{\frac{Z}{a}\sin w}{\left(\frac{Z}{a}\right)^2 - 2\left(\frac{Z}{a}\right)\cos w + 1}$$

$$= \frac{aZ \sin(w)}{Z^2 - 2aZ\cos w + a^2}$$

$$Z^2 + 0.1Z + 0.2 \equiv Z^2 - 2aZ\cos w + a^2$$

$$a^2 = 0.2 \Rightarrow a = \sqrt{0.2}$$

$0.1 = -2a \cos w$ ~~$= w$~~ $= 9$

$w = 96.4^{\circ} = 1.68$ rad

$\sin(w) = 0.99$

$$x(z) = \frac{4z\,(a \sin w)}{a \sin w\,[z^2 + 0.1z + 0.2]}$$

$$\Downarrow z^{-1}T$$

$$x(n) = \frac{4}{a \sin w} * a^n \sin(wn)$$

$$x(n) = 9\,(\sqrt{0.2})^n \sin(1.68n)$$

$\boxed{\text{Report}}$ $$x(z) = \frac{z}{z^2 + 4}$$

find $x(n)$

$\boxed{21}$ Lec 4